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Nonlinear noninertial response of a Brownian particle in a tilted periodic potential to a strong ac force

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The solution of the Langevin equation describing the dynamics of a Brownian particle in a tilted periodic potential in the overdamped limit is obtained in terms of a matrix continued fraction, allowing us to evaluate statistical averages governing the nonlinear response to a strong ac force. Pronounced nonlinear effects are observed for large values of the ac force. For a weak ac force and low noise strength, the results obtained agree closely with previously available linear response and noiseless solutions, respectively.

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The problem of the Brownian motion of a particle in a tilted periodic potential arises in a number of physical applications, for example, current-voltage characteristics of the Josephson junction [1], mobility of superionic conductors [2], a laser with injected signal [3], phase-locking techniques in radio engineering [4], dielectric relaxation of molecular crystals [5], etc. (This model currently merits attention in view of the intense interest in the effect of noise in the operation of nonlinear systems, e.g., stochastic resonance [6], and of the ever increasing areas of application of the model, e.g., to the ring-laser gyroscope [7].) A comprehensive discussion of the model is given in [8,9]. A concise method of numerical treatment of the model (in terms of infinite continued fractions) with a particular application to a ring-laser gyroscope has been suggested by Cresser *et al.* [10]. Further development of this approach has been given in Refs. [8–11]. However, all the solutions obtained in [8–11] are valid only for a weak ac external signal and so pertain to the *linear response*. Nevertheless, a variety of problems exists (e.g., the nonlinear impedance of a Josephson junction [1], the quantum noise effect on the mean beat frequency of a dithered-ring-laser gyroscope [12], etc.), where the *nonlinear response* to a strong ac force is required. The calculation of the ac nonlinear response is a difficult task as there is no longer *any* connection between the step-on and the step-off responses and the ac response because the response now depends on the precise nature of the stimulus—as no *unique* response function valid for all stimuli unlike the linear response exists. Attempts to calculate the nonlinear ac response of a Brownian particle in a tilted periodic potential have been made by many authors usually by means of the perturbation theory [5] so that the results are valid for low ac force amplitudes only, or in the noiseless limit, where the underlying nonlinear equation of motion can be solved numerically (see, e.g., [13,14] and references cited therein).

Here the ac nonlinear response of a Brownian particle in a tilted cosine potential in the presence of noise is evaluated exactly applying the matrix continued fraction technique commonly used in nonlinear response problems [15]. Our approach is a further development of those of Ref. [15] for the calculation of the harmonic mixing signal in a cosine potential and of Ref. [12] for the evaluation of the mean beat frequency of a dithered-ring-laser gyroscope. However, the approach used here differs from those of Refs. [12] and [15] (principally because the *time-dependent* portion of the ac nonlinear response may now be evaluated) and has the merit of being considerably simpler that those previously available (for example, the expressions obtained in Ref. [12] for the frequency-dependent dc portion of the response are so complicated that they are of limited use in practice). The stationary ac nonlinear response was not extensively addressed before as it was not of experimental interest until recently, e.g., to the nonlinear ac (microwave) impedance of intrinsic and fabricated Josephson junctions in the high temperature superconductors [14]. Thus, it is timely to accomplish a detailed study of the problem under consideration.

The Langevin equation of motion of a Brownian particle in a tilted cosine potential written in a dimensionless form is given by [8]

$$\frac{d^2}{dt^2}y(t)+\gamma\frac{d}{dt}y(t)+F_0\sin y(t)=F_{\mathrm{dc}}+F_m\cos\omega t+f(t). \tag{1}$$

where $f(t)$ is a white noise driving force such that

$$\overline{f(t)f(t')}=2\gamma\delta(t-t'),$$

$\delta(t)$ is the Dirac δ function, and the overbar means the statistical average over an ensemble of particles that have all started at time t with the same (sharp) initial position $y(t)=y$ and velocity $\dot{y}(t)=\dot{y}$. In the present case, we shall consider the overdamped limit only, which allows one to omit the inertial term $\ddot{y}$ in Eq. (1) [8]. This restricts the range of frequencies $(\omega\ll\sqrt{F_0})$ in which the model is applicable. Equation (1) now becomes [8]

$$\tau\frac{d}{dt}y(t)-x-\xi\cos\omega t+\sin y(t)=F_0^{-1}f(t), \qquad (2)$$

where $x=F_{\rm dc}/F_0$ and $\xi=F_m/F_0$ are the tilt and nonlinear parameters, and $\tau=\gamma/F_0$ is a relaxation time. Here we shall use the Stratonovich definition of a stochastic differential equation [16] as that definition is the mathematical idealization of the noninertial relaxation process under consideration [8,9]. On making the transformation $y\rightarrow r^n=e^{-iny}$ in Eq. (2), one obtains a stochastic differential equation with a multiplicative noise term, the averaging of which yields the differential-recurrence relations for the moments $\langle\overline{r^n}\rangle=\langle\overline{e^{-iny}}\rangle$ (as described in detail in Ref. [9]), viz.,

$$\tau\frac{d}{dt}\langle\overline{r^n}\rangle+[in(x+\xi\cos\omega t)+n^2/\gamma]\langle\overline{r^n}\rangle=\frac{n}{2}[\langle\overline{r^{n-1}}\rangle-\langle\overline{r^{n+1}}\rangle]. \qquad (3)$$

Here the angular brackets mean averaging over the sharp values of y at time t [9]. By assuming that the magnitude of the ac force parameter ξ in Eq. (3) may take an arbitrary value, one is faced with an intrinsically nonlinear problem, which can be solved as follows.

Since we are solely concerned with the stationary ac response, which is independent of the initial condition, one needs to calculate the solution of Eq. (3) corresponding to the stationary state. To accomplish this, one may seek all the $\langle\overline{r^n}\rangle$ in the form

$$\langle\overline{r^n}\rangle(t)=\sum_{k=-\infty}^{\infty}F_k^n(\omega)e^{ik\omega t}. \qquad (4)$$

On substituting Eq. (4) into Eq. (3), we obtain recurrence equations for the Fourier amplitudes $F_k^n(\omega)$, viz.,

$$F_k^{n+1}(\omega)+iz_{n,k}(\omega)F_k^n(\omega)+i\xi[F_{k-1}^n(\omega)+F_{k+1}^n(\omega)]-F_k^{n-1}(\omega)=0, \qquad (5)$$

where $z_{n,k}(\omega)=2(x+\omega\tau k/n-in/\gamma)$. The solution of the two variable $(k,\ n)$ recurrence Eq. (5) can be obtained in terms of matrix continued fractions as follows. Let us introduce infinite column vectors $\mathbf{C}_n(\omega)$ given by

$$\mathbf{C}_n(\omega)\begin{pmatrix}\vdots\\F_{-2}^n(\omega)\\F_{-1}^n(\omega)\\F_0^n(\omega)\\F_1^n(\omega)\\F_2^n(\omega)\\\vdots\end{pmatrix}\quad\text{with}\quad\mathbf{C}_0(\omega)=\mathbf{C}_0=\begin{pmatrix}\vdots\\0\\0\\1\\0\\0\\\vdots\end{pmatrix}.$$

Next, for $n\geqslant0$ the scalar *five-term* recurrence Eq. (5) can be transformed into the *matrix three-term* recurrence equation

$$\mathbf{Q}_n(\omega)\mathbf{C}_n(\omega)+\mathbf{C}_{n+1}(\omega)=\mathbf{C}_{n-1}(\omega),\quad n=1,2,3,..., \qquad (6)$$

where $\mathbf{Q}_n(\omega)$ is a tridiagonal infinite matrix given by

$$\mathbf{Q}_n(\omega)=i\begin{pmatrix}\ddots&\vdots&\vdots&\vdots&\vdots&\vdots&\iddots\\\cdots&z_{n,-2}(\omega)&\xi&0&0&0&\cdots\\\cdots&\xi&z_{n,-1}(\omega)&\xi&0&0&\cdots\\\cdots&0&\xi&z_{n,0}(\omega)&\xi&0&\cdots\\\cdots&0&0&\xi&z_{n,1}(\omega)&\xi&\cdots\\\cdots&0&0&0&\xi&z_{n,2}(\omega)&\cdots\\\iddots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

The recurrence Eq. (6) can be solved for $\mathbf{C}_1$ in terms of matrix continued fractions, viz.,

$$\mathbf{C}_1(\omega)=\cfrac{\mathbf{I}}{\mathbf{Q}_1(\omega)+\cfrac{\mathbf{I}}{\mathbf{Q}_2(\omega)+\cfrac{\mathbf{I}}{\mathbf{Q}_3(\omega)+\cdots}}}\mathbf{C}_0, \qquad (7)$$

where the fraction lines designate the matrix inversions and $\mathbf{I}$ is the identity matrix of infinite dimension. The column vector $\mathbf{C}_n(\omega)$ for $n=-1$ can also be obtained from Eq. (7) by taking into account that $F_0^{-1}(\omega)=F_0^{1*}(\omega)$ and $F_k^{-1}(\omega)=-F_{-k}^{1*}(\omega)$ for $k\neq0$ (the asterisk denotes the complex conjugate).

Having determined the column vectors $\mathbf{C}_1(\omega)$ and $\mathbf{C}_{-1}(\omega)$, one may evaluate from Eqs. (2) and (7) the nonlinear response $\langle V\rangle=\tau\langle\overline{\dot{y}}\rangle$ given by

$$\langle V\rangle=x+\xi\cos\omega t-\langle\overline{\sin y}\rangle(t), \qquad (8)$$

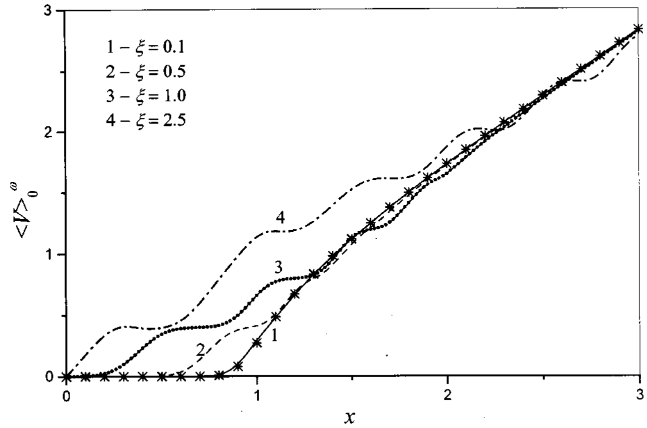

FIG. 1. $\langle V\rangle_0^\omega$ vs bias x parameter for various ξ ($\gamma=25$ and $\omega\tau=0.4$), showing the force-induced (Shapiro) steps; stars, Eq. (12).

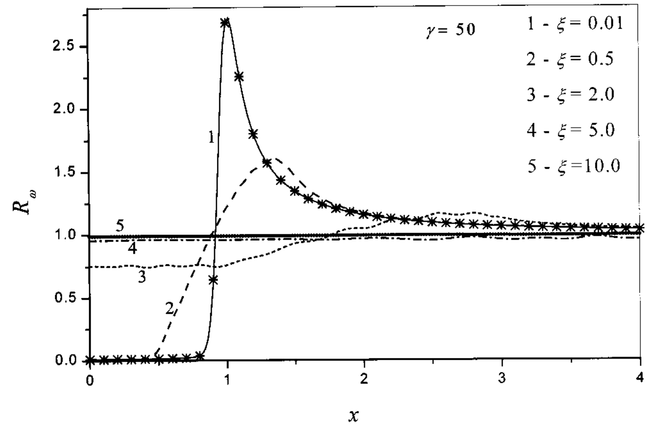

FIG. 2. R_ω vs x for various ξ ($\gamma=50$ and $\omega\tau=0.1$), showing pronounced departure from linear response (curve 1) as ξ is increased; stars, Eq. (13).

where $\langle\overline{\sin y}\rangle(t)=i[\langle\overline{r}\rangle-\langle\overline{r^{-1}}\rangle]/2$. The definition of Eq. (8) is very useful for particular applications as it determines, e.g., the current-voltage characteristics and nonlinear impedance of a Josephson junction [1], the mean beat frequency of a dithered-ring-laser gyroscope [7], etc. Thus, one can calculate both the *time-independent* (but frequency-dependent) dc response

$$\langle V\rangle_0^\omega=x+\mathrm{Im}[F_0^1(\omega)] \tag{9}$$

and the *time-dependent stationary* ac response

$$\langle V-\langle V\rangle_0^\omega\rangle=\sum_{k=1}^{\infty}\xi^k\,\mathrm{Re}[Z_k(\omega)e^{ik\omega t}], \tag{10}$$

where

$$Z_k(\omega)=\delta_{1,k}-i\xi^{-k}[F_k^1(\omega)+F_{-k}^{1*}(\omega)],$$

$\delta_{i,k}$ is Kronecker's delta. The limit of a weak ac force ($\xi\ll1$) allows us to calculate from Eqs. (9) and (10) the linear response to an ac force $F_m e^{i\omega t}$ as well. On noting that $\langle V\rangle=x-\langle\sin y\rangle_0+\langle V\rangle_1$, where the subscripts "0" and "1" denote the average in the absence of the ac force and the average which is linear in $F_m e^{i\omega t}$, respectively, we have

$$\langle V\rangle_1=Z_1(\omega)\xi e^{i\omega t}. \tag{11}$$

The matrix continued fraction solution just obtained is a general result for the nonlinear response to an ac external force of arbitrary amplitude. It may be shown by means of the direct numerical calculation that the matrix continued fraction in Eq. (7) converges in all ranges of the model parameters of interest (algorithms for calculating matrix continued fractions are discussed in Ref. [8], Chap. 9).

Some results of the calculation of nonlinear response from Eqs. (7)–(10) are shown in Figs. 1–3. In Fig. 1, $\langle V\rangle_0^\omega$ versus the bias parameter x is shown for various amplitudes of the ac force ξ and noise strength γ (the limit $\gamma\to\infty$ corresponds to the noiseless limit). As is apparent from these figures, for low noise and high frequencies the shape of the characteristics becomes distorted when ξ and γ increase and the steps induced (for the Josephson junction they are known as Shapiro steps) on the curves adhere to the line $\langle V\rangle_0^\omega=x$. For $\xi=0$, our theory yields the frequency-independent dc response $\langle V\rangle_0$, which is in agreement with that of Ambegaokar and Halperin [17], who computed the dc current-voltage characteristic of a Josephson junction by calculating the time-independent solution of the noninertial Fokker-Planck equation associated with the Langevin equation (2). Their results may be written as [9]

$$\langle V\rangle_0=x+\mathrm{Im}[I_{1+ix\gamma}(\gamma)/I_{ix\gamma}(\gamma)]. \tag{12}$$

The in-phase $R_\omega=\mathrm{Re}[Z_1(\omega)]$ and the out-of-phase $X_\omega=-\mathrm{Im}[Z_1(\omega)]$ parts of the first harmonic dynamic response versus x for various ξ are presented in Figs. 2 and 3, showing strong phase-locking effects in the ac response, similar to the steps seen in $\langle V\rangle_0^\omega$. For large values of the dc bias the influence of the ac force diminishes and R_ω and X_ω approach unity and zero, respectively. Figures 2 and 3 show clearly that the response saturates at large ξ. For $\xi\ll1$, the present theory is in accordance with the results of the calculation of the linear response described in Refs. [9] and [11] for the particular application to the Josephson junction, where both exact and approximate equations for the linear impedance $Z_1(\omega)$ of the junction have been derived. In our notation the approximate equation for $Z_1(\omega)$ reads

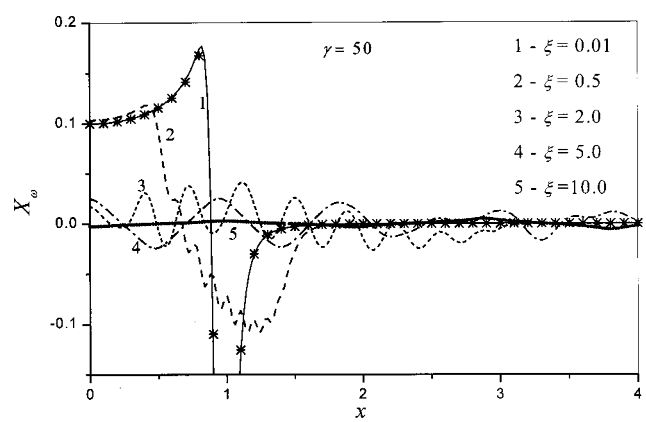

FIG. 3. The same as in Fig. 2 for X_ω.

$$Z_1(\omega)=1-\frac{1}{2}\left[\frac{I_{1+ix\gamma}(\gamma)/I_{ix\gamma}(\gamma)}{\lambda-i\omega\tau}+\frac{I_{1-ix\gamma}(\gamma)/I_{-ix\gamma}(\gamma)}{\lambda^*-i\omega\tau}\right], \tag{13}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind of order ν [18] and λ is given by

$$\lambda=\frac{I_{ix\gamma}(\gamma)I_{1+ix\gamma}(\gamma)}{2\int_0^\gamma I_{ix\gamma}(t)I_{1+ix\gamma}(t)dt}.$$

Figures 2 and 3 clearly demonstrate that Eq. (13) yields perfect correspondence to the exact solution for $\xi \ll 1$ and that the nonlinear response has a striking departure from linear response as ξ is increased. For $\gamma \gg 1$, the nonlinear effects become more pronounced and the results are in agreement with those of Ref. [14] for the Josephson junction nonlinear impedance in the noiseless limit.

Here we have presented a usable method of analysis of the nonlinear response of a Brownian particle in a tilted periodic (cosine) potential in the presence of noise for wide ranges of the nonlinear (ξ), the noise strength (γ), and the bias (x) model parameters. The results are valid for low driving frequencies $\omega \ll \omega_p$, where $\omega_p=\sqrt{F_0}$, since we have ignored the inertial term in Eq. (1). In order to obtain them we have used the matrix continued fraction approach, which allowed us to solve the problem exactly. We have shown how pronounced nonlinear effects appear in the response for low noise ($\gamma \gg 1$) and frequencies ω satisfying the condition $\gamma^{-1} \leqslant \gamma\omega/F_0 \leqslant 1$. The linear response solutions are recovered from the nonlinear response in the weak ac force limit. The theoretical results presented here may be useful for the interpretation of experimental data for the microwave resistance and reactance of superconducting weak links. For example, nonlinear effects in the microwave resistance of a high quality single crystal $YBa_2Cu_3O_{7-\delta}$ were observed experimentally in [19], where it was shown that the resistively shunted junction (RSJ) model, which is a particular case of the present model, describes the essential features of the nonlinear microwave surface impedance Z_s. Experimental data on Z_s of high temperature superconducting thin films were also successfully interpreted in the context of the RSJ model [20]. However, theoretical estimates in [14,19,20] were carried out in the noiseless limit only. Our approach can also be applied to the analysis of the nonlinear impedance taking account of noise effects so as to provide a better quantitative agreement with experiments. We reiterate that a Langevin equation of the kind used here also arises in a number of other stochastic systems with a cosine potential subjected to a strong ac driving force. Therefore the present results may also be applied to the evaluation of the nonlinear response of such systems. The application of the approach to particular nonlinear response problems will be published elsewhere.

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[1] G. Barone and A. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).
[2] W. Dieterich, P. Fulde, and I. Peschel, Adv. Phys. **29**, 527 (1980).
[3] W. W. Chow, M. O. Scully, and E. W. Van Stryland, Opt. Commun. **15**, 6 (1975).
[4] A. J. Viterbi, *Principles of Coherent Communication* (McGraw-Hill, New York, 1966).
[5] W. T. Coffey, Adv. Chem. Phys. **63**, 69 (1985).
[6] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[7] W. W. Chow, J. Gea-Banacloche, L. M. Pedrotti, V. Sanders, W. Schleich, and M. O. Scully, Rev. Mod. Phys. **57**, 61 (1985).
[8] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).
[9] W. T. Coffey, Yu. P. Kalmykov, and J. T. Waldron, *The Langevin Equation* (World Scientific, Singapore, 1996), Chap. 6.
[10] J. D. Cresser, D. Hammonds, W. H. Louisell, P. Meystre, and H. Risken, Phys. Rev. A **25**, 2226 (1982).
[11] W. T. Coffey, Yu. P. Kalmykov, and E. S. Massawe, Phys. Rev. E **48**, 77 (1993).
[12] W. Schleich, C.-S. Cha, and J. D. Cresser, Phys. Rev. A **29**, 230 (1984).
[13] J. McDonald and J. R. Clem, Phys. Rev. B **56**, 14 723 (1997).
[14] Z. Zhai, P. V. Parimi, and S. Sridhar, Phys. Rev. B **59**, 9573 (1999).
[15] H.-J. Breymayer, H. Risken, H. D. Vollmer, and W. Wonneberger, Appl. Phys. B: Photophys. Laser Chem. **28**, 335 (1982).
[16] R. L. Stratonovich, *Conditional Markov Processes and Their Application to the Theory of Optimal Control* (Elsevier, New York, 1968).
[17] V. Ambegaokar and B. I. Halperin, Phys. Rev. Lett. **22**, 1364 (1969).
[18] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), p. 46.
[19] Z. Zhai, H. Srikanth, S. Sridhar, A. Erb, E. Walker, and R. Flukinger, Physica C **282–287**, 1601 (1997).
[20] B. B. Jin and R. X. Wu, J. Appl. Phys. **84**, 3250 (1998).